Convergence of Ritz values in the isometric Arnoldi process

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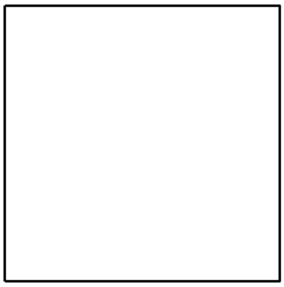
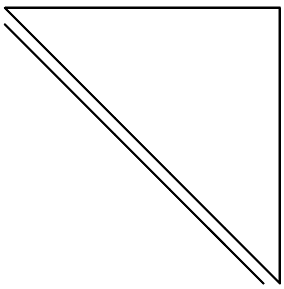
The IAP



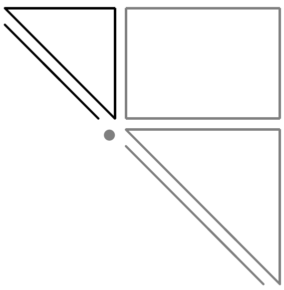
full unitary matrix

↓



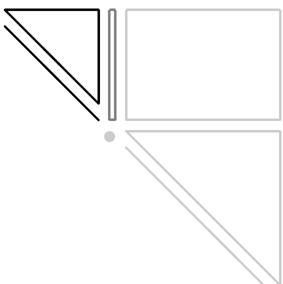
reduction to Hessenberg form;
the subdiagonal contains strictly positive numbers

↓



taking a principal left upper block;
this is not unitary anymore

↓



renormalizing the last column to obtain a unitary matrix

Setting

- A *sequence* of unitary matrices $(U_N)_N$, each of dimension $N \times N$.
- The eigenvalues $\{\lambda_{k,N}\}_k$ and orthonormal eigenvectors $\{v_{k,N}\}_k$.
- A unit starting vector $b_N \in \mathbb{C}^N$ for every N.
- The $n \times n$ unitary Hessenberg matrix $H_{n,N}$ created by n steps of the IAP, with $n < N$.
- The characteristic polynomials $\psi_{n,N}$ of $H_{n,N}$.
- The eigenvalues $\{\theta_{k,n,N}\}_k$ of $H_{n,N}$, which are called the isometric Ritz values.

Potential Theory

$$U^\mu(z) = \int \log \frac{1}{|z - z'|} \, \mathrm{d}\mu(z'),$$

$$I(\mu) = \iint \log \frac{1}{|z - z'|} \, \mathrm{d}\mu(z)\mathrm{d}\mu(z').$$

Conditions

1. There exists a probability measure σ with U^σ real valued and continuous such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_{j,N}} = \sigma.$$

2. For all $\varepsilon > 0$ there exists a $\delta \in (0,1)$ and an $N_0 \in \mathbb{N}$ so that for all $N > N_0$ and for all $k \leqslant N$

$$\prod_{\substack{j=1 \\ 0<|\lambda_{j,N}-\lambda_{k,N}|<\delta}}^{N} |\lambda_{j,N} - \lambda_{k,N}| > \mathrm{e}^{-N\varepsilon}.$$

3. For every N, we have that $\|b_N\| = 1$ and

$$\lim_{N\to\infty} \Big(\min_{1\leqslant k\leqslant N} |\langle b_N, v_{k,N}\rangle| \Big)^{1/N} = 1.$$

Results

Theorem 1 *There exists a probability measure μ_t, depending only on t and σ, such that*

$$\lim_{\substack{n,N\to\infty \\ n/N\to t}} \frac{1}{n}\sum_{j=1}^{n} \delta_{\theta_{j,n,N}} = \mu_t,$$

$$0 \leqslant t\mu_t \leqslant \sigma, \qquad \int \mathrm{d}\mu_t = 1.$$

μ_t minimizes the logarithmic energy $I(\mu)$ among all measures μ satisfying $0 \leqslant t\mu \leqslant \sigma$ and $\int \mathrm{d}\mu = 1$. Moreover the logarithmic potential U^{μ_t} of μ_t is a continuous function on $\mathbb{C}$. There also exists a real constant F_t such that

$$\lim_{\substack{n,N\to\infty \\ n/N\to t}} \|\psi_{n,N}(U_N)b_N\|^{1/n} = \exp(-F_t)$$

and

$$\begin{cases} U^{\mu_t}(z) = F_t & \text{for } z \in \operatorname{supp}(\sigma - t\mu_t), \\ U^{\mu_t}(z) \leqslant F_t & \text{for } z \in \mathbb{C}. \end{cases}$$

Results

Theorem 2 *For every* $(\lambda_{k_N,N})_N$ *converging to* $\lambda \in \mathbb{T}$ *and for every* $t \in (0,1)$

$$\limsup_{\substack{n,N\to\infty\\ n/N\to t}} \min_j |\lambda_{k_N,N} - \theta_{j,n,N}|^{1/n} \leqslant \exp\Big(U^{\mu_t}(\lambda) - F_t\Big).$$

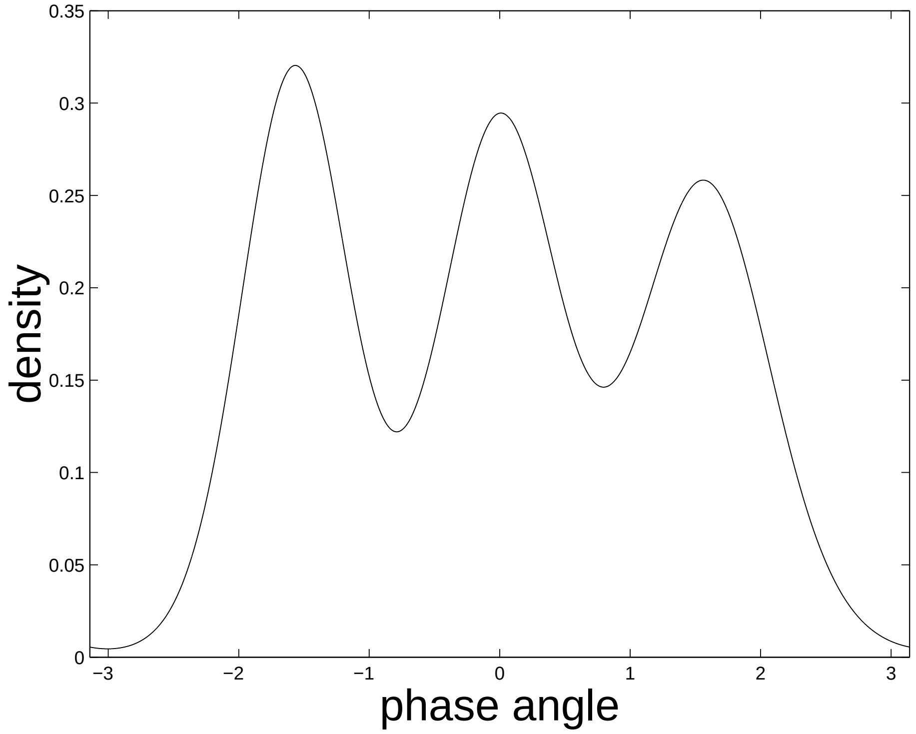
We define the set

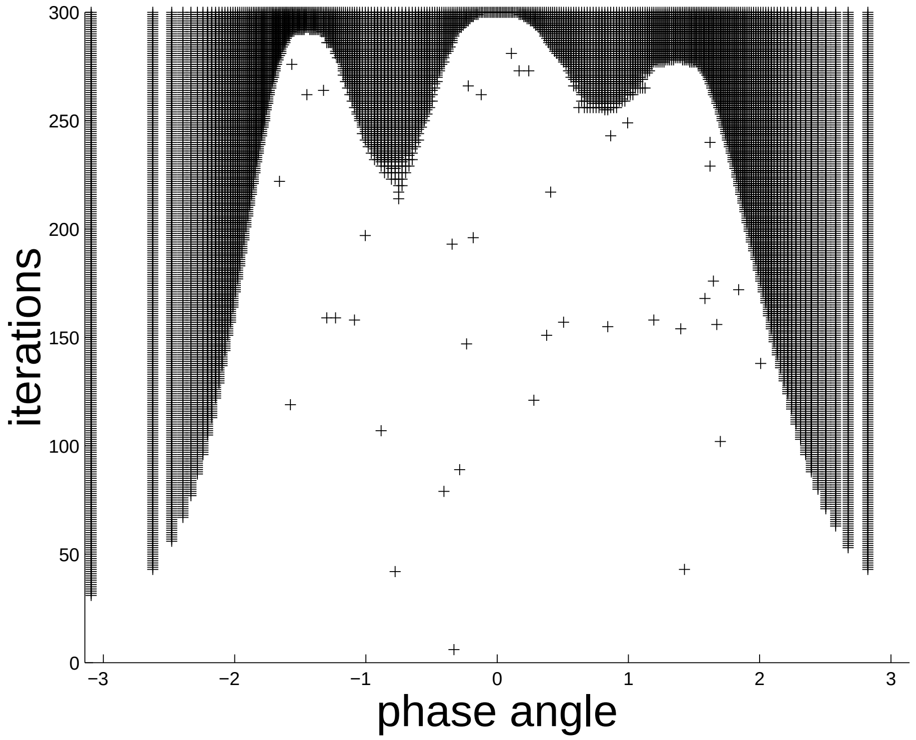
$$\Lambda(t,\sigma) := \{z \in \mathbb{T} \mid U^{\mu_t}(z) < F_t\}.$$

Theorem 3 *For* nearly *every* $(\lambda_{k_N,N})_N$ *converging to* $\lambda \in \Lambda(t,\sigma)$ *and for every* $t \in (0,1)$

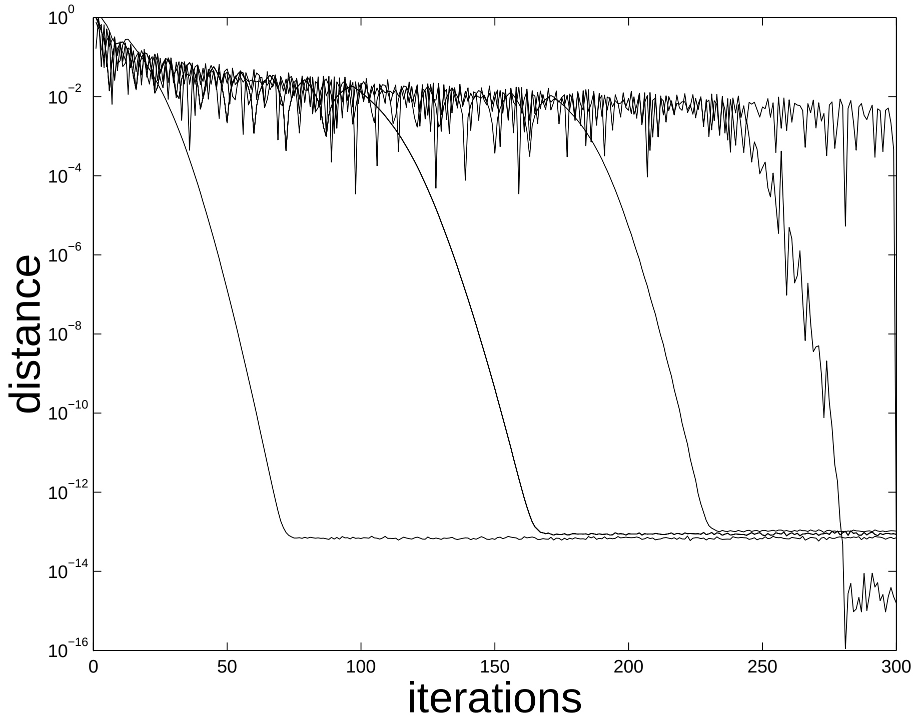
$$\lim_{\substack{n,N\to\infty\\ n/N\to t}} \min_j |\lambda_{k_N,N} - \theta_{j,n,N}|^{1/n} = \exp\Big(2\big(U^{\mu_t}(\lambda) - F_t\big)\Big).$$

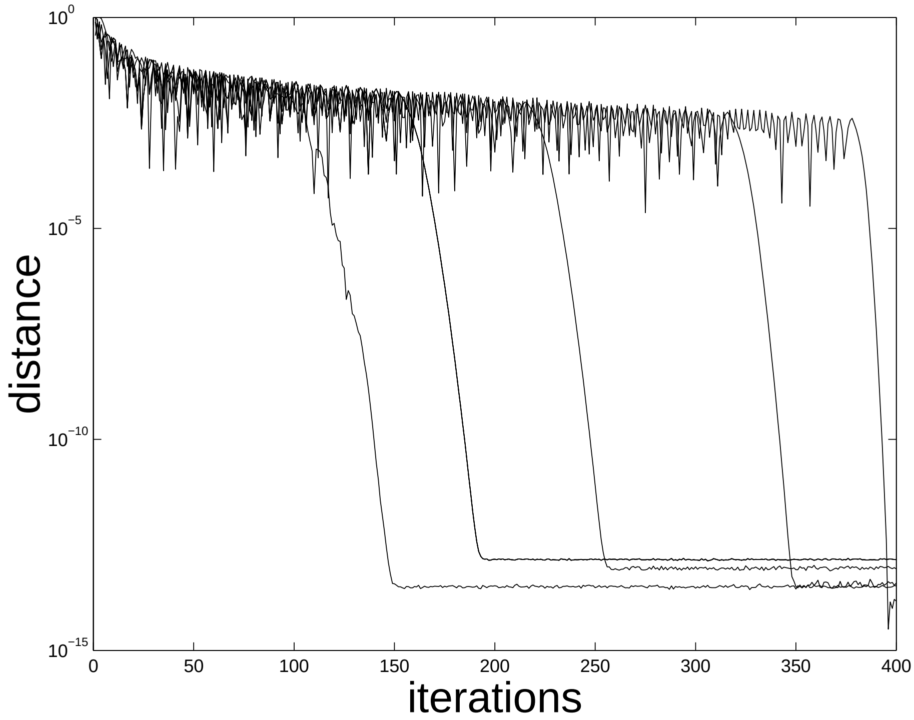
Numerical Experiments





Numerical Experiments





Positioning i.R.v.